Theorem 1. *Let $[a], [b] \in \mathbb{Z}_n$. Then $[a] \cdot [b] = [0]$ implies one of $[a]$ or $[b]$ is $[0]$ if and only if n is prime.*

Proof.

($\Longrightarrow$) Suppose that n is composite. Then $n = xy$ with $1 < x, y < n$. Thus $[x], [y] \neq [0]$, but $[x][y] = [xy] = [n] = [0]$. This proves the forward direction by contrapositive.

($\Longleftarrow$) Suppose that n is prime and suppose that $[a][b] = [0]$. Without loss of generality, assume that $[a] \neq 0$. Let $x \in [a]$ and $y \in [b]$ such that $0 < x, y \leq n$. Since $[a] \neq [0]$, we know that $0 < x < n$. Since $[a][b] = [ab] = [xy] = [0]$, we have that $xy \equiv 0 \mod n$, i.e., $xy = nz$ for some $z \in \mathbb{Z}$. Since n is prime, $1 < x < n$, and $x | nz$, it follows that $x | z$ since $x \nmid n$. Thus $z = kx$ for some $k \in \mathbb{Z}$ and substituting that into the earlier equation, we get

$$xy = nz = nkx.$$

Since $x \neq 0$, we then have

$$y = nk.$$

This proves that $[b] = [y] = [nk] = [0]$.

$\square$